The regression of atmospheric concentration on carbon dioxide emissions

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Key Points:

- The impulse response of atmospheric CO2 concentration to variations in emissions can be estimated from observations.
- It decays to zero at infinite lag with no remnant fraction.
- The remnant fraction predicted by circulation models is due to their failure to account for eddy diffusion caused by deep ocean turbulence.

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Abstract

The impulse response of CO2 concentration due to variation in CO2 emissions was estimated from observed time series using the ARX method augmented by testing residuals for self-correlation. It was found to be a simple exponential with a half-time of 43 years with no remnant component. The longer half times and large remnant fraction of the impulse response derived from ocean circulation models is attributed to the failure of these models to account for turbulent mixing in the deep ocean.

Plain Language Summary

When using climate models to make projections of future climate it is important to know how the atmospheric concentration of CO2 will change in response to changes in carbon emissions. The "impulse response" of concentration due to emissions summarises the required information in a single curve from which the response to any arbitrary emissions scenario can be easily determined. It is a curve showing how the concentration will change following a single short impulse in emissions. The impulse responses widely accepted by the modelling community all have a "remnant fraction" of between 10 and 20 percent implying that this fraction of emitted CO2 remains in the atmosphere forever. These curves are, themselves, based on circulation models similar to climate models. This paper develops a statistical technique for estimating the impulse response directly from the data while making no assumptions about the underlying physics. The impulse response estimated in this way shows that CO2 remains in the atmosphere for a shorter time than hitherto supposed and has no remnant fraction. All the CO2 presently generated by fossil fuel will ultimately leave the atmosphere; half will be gone in the next half century.

1 Introduction

It is common practice across a wide range of sciences to treat physical quantities as ensemble parameters and to estimate them from sample statistics. A time series is a particularly type of sample, one in which a series of measurements are taken at equal intervals of time or averaged over equal intervals of time. The Pearson correlation coefficient is often used to describe the relationship between contemporaneous time series, but it is a poor statistic because it does not account for temporal ordering. Two other statistics, which better summarize the relationship between two concurrent time series, are the impulse response and the sensitivity. The impulse response is the response of the endogenous or dependant variable to a short pulse in the exogenous or independent variable. The sensitivity is defined here as the response of the endogenous variable to unit step-function in the exogenous variable. It is the sum of terms (or integral) of the impulse response.

Both statistics can be estimated using the "autoregressive with exogenous variable" or ARX method. Their existence and the number of ARX regression coefficients required for their computation, is established by rejecting those configurations whose residuals show significant self-correlation. The impulse response is then found as the convolutional reciprocal of a sequence derived from the ARX regression coefficients. These ideas originated with Box and Jenkins (1976). Here we derive these statistics using convolutional methods and apply them to carbon cycle time series.

The present statistical method makes no assumptions about the underlying physics, other than that the system under investigation is random, rather than deterministic, and that it is stationary and ergodic. It gives results which conflict with those based on numerical models.

2 The Model-Derived Carbon Cycle Impulse Response Function

A normalized Impulse Response Function $\Im(t)$ was first derived using a global circulation model by Maier-Reimer and Hasselmann (1987) (MRH), viz:

$$\Im(t) = A_0 + \sum_{j=1}^{4} A_j \exp(-t/\tau_j) \quad (1)$$

Where the A_j are the proportions corresponding to various decay times, τ_j. A_0 is non-zero. The time constants, τ_j range from 1.2 years to 362.9 years. A very similar model, the HILDA model, was proposed by Siegenthaler and Joos (1992) which ultimately became the Bern model of the IPCC reports. The impulse response function, once known, is a great convenience for climate modellers because it allows atmospheric CO_2 concentration, $y(x)$, to be determined for any arbitrary emission rate, $x(t)$, using the convolution

$$y(x) = \int \Im(t - t')x(t')dt' \quad (2)$$

where $x(t)$ has been scaled to have the same units as $y(t)$.

In order to assess the non-linear response of pCO_2 to total carbon in the mixed layer, MRH ran their model using test input emission signals comprising increases of 25% , doubling and quadrupling of the initial atmospheric CO_2 concentration[1]. The three impulse response functions are shown in Figure 1. Values of A_0 were 0.131, 0.142 and 0.166 respectively which determine the remnant fractions of atmospheric CO_2 under the three scenarios.

There is something very odd about this. Certainly we might expect a remnant fraction to remain in the atmosphere once the oceanic reservoir is saturated. What is odd is that the three remnant fractions are almost the same. In each case, we would expect the reservoir to take up roughly the same *absolute* amount of CO_2 before it becomes saturated, in which case a much larger *fraction* would remain in the atmosphere in the quadrupling case than in the 25% increase case. This is not shown as happening to the dotted curves in Figure 1. The similarity of the remnant fractions in the three cases does not imply saturation. Rather, it implies a partitioning of the available CO_2 between two reservoirs with a volume ratio of the order of $(1-r)/r$, where r is the remnant fraction. When we apply this to the MRH model, the oceanic reservoir into which atmospheric CO_2 is diffused has only about six times the CO_2 capacity of the atmosphere. Given that the ocean has been estimated to carry fifty times the steady-state, atmospheric load of CO_2 (Houghton et al., 2001), this is a remarkably small value. It implies that, in the MRH model, CO_2 is partitioned between the atmosphere and a small "sub-reservoir" from which little escapes into the remainder of the ocean, a sub-reservoir roughly comprising the moving water mass that constitutes the "conveyor belt" of the thermohaline circulation.

[1] Their definition is slightly ambiguous. Their equation (5) defines it correctly but the accompanying text appears to define a "step response function".

3 The ARX Method

For notational convenience, in the following, all sample means have been removed and random variables are assumed to have zero mean.

The autoregressive moving average method with a single exogenous variable, ARMAX(p,q), is given at time, i, by:

$$Y_i = \alpha_0 x_i + \sum_{j=1}^{p} \alpha_j . y_{i-j} + \sum_{k=1}^{q} \beta_j \Xi_{i-k} \quad , \quad i = 1, ..., N \tag{3}$$

where the dependent random variable is Y_i, x_i is the exogenous variable, the y_i are past values of Y_i and the Ξ_i are unselfcorrelated random variables with zero mean. The regression coefficients α_0, α_j and β_j are estimated from the data and p and q are small positive integers. The notation is intended to make a clear distinction between random variables which are Latin upper case, and constants, such as past values of random variables, which are Latin lower case. Equation (3) is a state space representation (Hamilton, 1994) describing states of the system at a succession of discrete instants; the random variable, Y_i, at one instant becomes the constant, y_i, in the following instant. The direction of time is important in regression, which, unlike correlation, allows causality to be inferred (Granger, 1969).

There are software packages for parameter estimation available under the aegis of the major programming languages. Unfortunately some of these are flawed, because they estimate the exogenous parameter, α_0, prior to estimating the other parameters, leading to omitted-variable bias (Greene, 2003); all parameters must be estimated simultaneously in a regression model.

Estimation of the MA coefficients, $\{\beta_i\}$, requires an iterative Kalman filter method which may not converge. The second, moving average summation in (3), describes a convoluting or "blurring" function, so that $q > 1$ when the sampling interval, Δt, is too small. Estimation of the MA coefficients can be avoided by decimating the time series by q to give a new time series with a larger sampling interval, $q\Delta t$, for which the innovation sequence, $\{\Xi_m\}$, is unselfcorrelated. Then (3) becomes

$$Y_m = \alpha_0 x_m + \sum_{n=1}^{p} \alpha_n . y_{m-n} + \Xi_m \quad , \quad m = 1, ..., M \tag{4}$$

where $m = qi$, $qM \leq N$, The model summarized by (4) is an ARX(p) model for 'autoregressive with exogenous variable'. The regression coefficients, α_i, and their confidence limits are estimated using Ordinary Least Squares. The sequence of residuals, $\{\xi_m\}$, is given by

$$\xi_m = y_m - \left(\hat{\alpha}_0 x_m + \sum_{n=1}^{p} \hat{\alpha}_n . y_{m-n} \right) \quad , \quad m = 1, ..., M \tag{5}$$

where y_m is the sample value or 'realization' of Y_m and $\hat{\alpha}_0$ to $\hat{\alpha}_p$ are the regression coefficient estimates. The $\{\xi_m\}$ are tested using the Ljung-Box, Q statistic with probability P (Ljung & Box, 1978). The minimum number of coefficients, $\hat{p}$, is found for which P is greater than some confidence level, say 0.1, for which it can be assumed the innovation sequence is not self-correlated.

Our best estimate of the relationship between the two time series is then

$$\sum_{n=0}^{\hat{p}} \hat{\gamma}_n y_{m-n} = \hat{\alpha}_0 x_m \tag{6}$$

where

$$\hat{\gamma}_0 = 1 \tag{7}$$

and

$$\hat{\gamma}_n = -\hat{\alpha}_n \quad , \quad n = 1, ..., \hat{p} \tag{8}$$

The sequence $\{\gamma_n\}$ specified by (6) is the prediction error filter of the autoregressive process.

We can define a time series more precisely as a finite or semi-infinite sequence, $\{x_0, x_1, x_2, ...\}$ for which the index specifies successive equally spaced intervals of time. The convolution, $c = \{c_k; k = 0, 1, ..., r\} = a * b$, of two time series $a = \{a_i; i = 0, 1, ..., p-1\}$ and $b = \{b_j; j = 0, 1, ..., q\}$, is defined by

$$c_k = \sum_{i+j=k} a_i b_j \tag{9}$$

Under this definition convolution satisfies the commutative, associative and distributive laws of arithmetic. Note also that

$$\sum_i a_i \cdot \sum_j b_j = \sum_k \sum_{i+j=k} a_i b_j = \sum_k c_k \tag{10}$$

The sum on the left hand side of (6) is a convolution and (6) can be written

$$\hat{\gamma} * y = \hat{\alpha}_0 x \tag{11}$$

A more useful form of (11) is

$$y = \hat{I} * x \tag{12}$$

where $\hat{I}$ is the convolutional reciprocal of $\hat{\gamma}/\hat{\alpha}_0$ given by

$$\hat{I} * \hat{\gamma} = \hat{\alpha}_0\{1\} \tag{13}$$

and is termed the impulse response. It can be estimated numerically by iteration using (13) in the form

$$\hat{I}_m = \sum_{i=1}^{p} \hat{\alpha}_i \hat{I}_{m-i} + \hat{\alpha}_0 \delta_m \tag{14}$$

For display purposes and inter-comparison a normalized impulse response, $\Im$, may be used where

$$\Im * \gamma = \{1\} \tag{15}$$

Thus the normalized impulse response is the convolutional inverse of the prediction error filter of the autoregressive process.

Like the regression coefficients, I is a property of the system under investigation and $\hat{I}$ is its estimate. Equation (12) describes the output of the system, y, in response to *any* input sequence, x.

The sensitivity of the system, S, is defined here as the response at infinity to a unit step function, H_j, where $H_j = 0$ for $j < 0$ and $H_j = 1$ for $j \geq 0$. From (9)

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{i+j=k} I_i H_j = \sum_{k=0}^{\infty} I_k \tag{16}$$

i.e. it is the sum of the terms of the impulse response. It is a random variable on which confidence limits can be placed.

According to (10) the sum of a convolution is equal to the product of the sums of the convoluting factors. Thus, from (13)

$$\sum_{m=0}^{\infty} I_m \sum_{n=0}^{p} \gamma_n = S \sum_{n=0}^{p} \gamma_n = \alpha_0 \tag{17}$$

from which $\hat{S}$ can be estimated in terms of the prediction error filter, $\{\hat{\gamma}_n\}$.

4 Estimating the Impulse Response from Observations

The ARX method developed above was applied to annual means (Meinshausen et al., 2017) of atmospheric CO_2 concentration , C_i, vs global fossil fuel emissions, E_i. Global fossil fuel emissions for the interval 1850 to 2014, were downloaded from the Carbon Dioxide Information Analysis Center (Boden et al., 2017).

Applying the Ljung-Box test to the residuals given by (5) for ARX(p) for $p = 0, ..., 5$ resulted in zero probabilities in all cases. The ARMAX method revealed a significant moving average component with $q = 2$. For this reason both time series were decimated by 2 and the ARX / Ljung-Box method reapplied. The results for the decimated data are shown in Table 1. The probability, P, for the ARX(1) run has a value of 0.4359 indicating that the null hypothesis that the residuals are unselfcorrelated cannot be rejected. Thus the simplest regression relationship between C_i and E_i which unambiguously fits the data is the ARX(1) model, viz.:

$$C_i = \hat{\alpha}_0 E_i + \hat{\alpha}_1 C_{i-1} \ \ , \ \ i = 1, ..., N \tag{18}$$

where the estimated regression coefficients are $\hat{\alpha}_0 = 0.21$ and $\hat{\alpha}_1 = 0.969$ with 95 percent confidence limits 0.945 and 0.992. The prediction error filter is $\{1, -\hat{\alpha}_1\}$ which has convolutional inverse $\{1, \hat{\alpha}_1, \hat{\alpha}_1^2, ...\}$, a geometric sequence with common ratio $\hat{\alpha}_1$. The nth term of the IRS estimate is given by

$$\hat{I}_n = \hat{\alpha}_0 \hat{\alpha}_1^n = \hat{\alpha}_0 \exp(-\frac{nq\Delta t}{\tau}) \tag{19}$$

and the impulse response can be regarded as discretely sampled from a continuous exponential function with time constant given by

$$\tau = -q\Delta t \ln(\hat{\alpha}_1) \tag{20}$$

Substituting $\hat{\alpha}_1$ and its confidence limits into (20) and multiplying by ln(2) gives a half-time of 43 years with confidence limits of 24 years and 193 years. The normalized impulse response is shown in Figure 1.

The sensitivity estimate, S, was 6.77 p.p.m.$GtCO_2^{-1}$.year with 95 percent, t-test confidence limits of 4.03 and 20.15 p.p.m.$GtCO_2^{-1}$.year. The probability that $S > 10^5$ was 0.012.

5 Discussion

The impulse response and sensitivity of CO_2 concentration estimated here are quite different from conventionally accepted values. The impulse response has an exponential decay with a single time constant and the sensitivity estimate is finite. These statistics were derived from an ARX regression model which was an excellent fit to the observations with no self-correlation evident in the residuals

The impulse response function (1) due to Maier-Reimer and Hasselmann (1987) must be integrated from zero to infinity with respect to time in order to give the sensitivity. Their sensitivity is therefore infinite because of a non-zero constant in (1) according to which 13 percent of CO_2 emissions remains in the atmosphere indefinitely. In contrast the sensitivity estimated here from observations is finite, implying a serious shortcoming of their dynamical global ocean circulation model.

A possible explanation is the following. The deep ocean is bounded by a turbulent mixed layer and by the highly turbulent Antarctic Circumpolar Current. It is therefore likely to be internally mixed by a Kolmogorov cascade of turbulent eddies, some with spatial scales as large as ocean basins and with time scales of, perhaps, decades. Turbulence is a stochastic phenomenon which is difficult to observe at large spatial and temporal scales and which cannot be readily emulated by deterministic models. The complexity of the eddy transports noted by Kamenkovich et al. (2021) calls for reconsideration of how they are estimated in practice, particularly in general circulation models. Eddy diffusion generated by such eddy transports would greatly increase the capacity of the deep ocean to absorb carbon dioxide and so would account for the shorter half time of the observed impulse response of atmospheric CO_2 concentration. Whatever the explanation, there is no observational evidence for the long half times and remnant component of atmospheric CO_2 concentration presently assumed by most modellers.

6 Open Research

Software and data used in the preparation of this article are available for download at Zenodo under the heading *Flawed Carbon Cycle Models*: DOI: 10.5281/zenodo.6302014 (`https://zenodo.org/record/6302014#.YhwtrjpxXct`)

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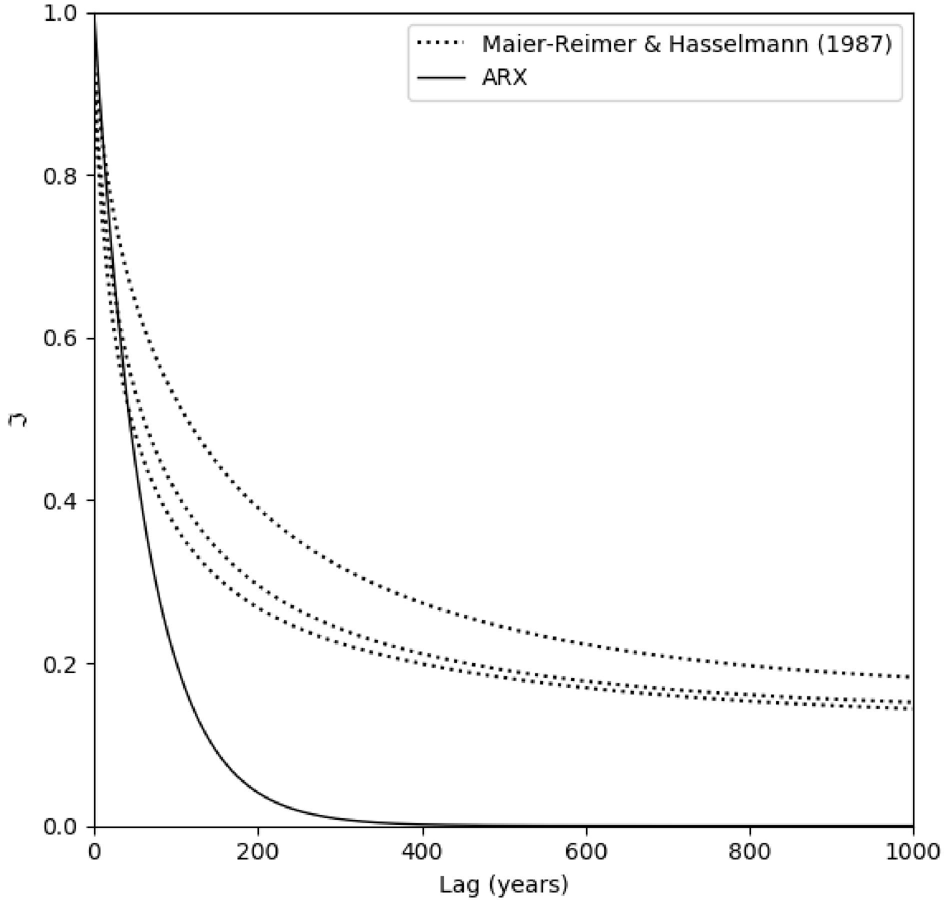

Figure 1. The normalized impulse response, $\Im$, of Carbon Dioxide concentration due to an impulse in CO2 emissions derived from observed time series using the ARX method (solid line). Also shown are the model-derived, normalized impulse response functions of Maier-Reimer and Hasselmann (1987) (dotteded lines).

Run	Q	pvalue
C(t) vs E(t) only	513.5	0.0000
C(t) vs E(t), C(t-1)	28.5	0.4359
C(t) vs E(t),C(t-1),C(t-2)	28.6	0.3830
C(t) vs E(t) to C(t-3)	24.5	0.5483
C(t) vs E(t) to C(t-4))	24.3	0.5049
C(t) vs E(t) to C(t-5)	22.0	0.5796

Table 1. Ljung-Box parameter, Q, and its probability, P, for five ARX runs of CO_2 concentration, C, vs. global fossil fuel emissions, E. Both time series were decimated by 2.